Chapter 3B

Discrete Random Variables

Random Variables

- A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

- Notation:
  - An *rv* is typically denoted by an uppercase letter, such as X.
  - After the data s collected, the measured value is denoted by a lowercase letter, such a $x = 70$. X and x are usually shown in italics, e.g., $P(X=x)$.

Continuous vs. Discrete RVs

- A discrete random variable is a *rv* with a finite (or countably infinite) range. They are usually integer counts.
  - Number of scratches on a surface.
  - Proportion of defective parts among 100 tested.
  - Number of transmitted bits received in error.

- A continuous random variable is a *rv* with an interval (either finite or infinite) of real numbers for its range. Its precision depends on the measuring instrument.
  - Electrical current and voltage.
  - Physical measurements, e.g., length, weight, time, temperature, pressure.

Some DRV examples

- A phone system for a business contains 48 lines. Let X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48.
  - The system is observed at a random time. If 10 lines are in use, then $x = 10$.
- Define the random variable X to be the number of contamination particles on a wafer.
  - The possible values of X are the integers 0 through a very large number, so we write $x \geq 0$.
- We could also describe the random variable Y as the number of chips made from a wafer that fail a final test.
  - If there can be 12 chips made from a wafer, then we write $0 \leq y \leq 12$.

Discrete Probability Distributions

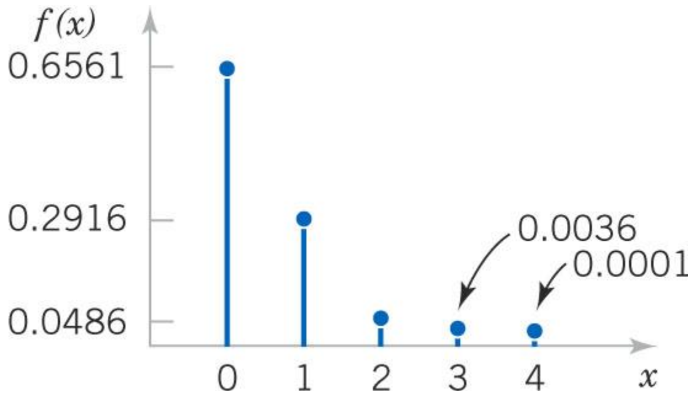
- A random variable *X* associates the outcomes of a random experiment to a number on the number line.

- The probability distribution of the random variable *X* is a description of the probabilities associated with the possible numerical values of *X*.

- A probability distribution of a discrete random variable can be:
  - A table or list of the possible values along with their probabilities.
  - A graph from that table.
  - A formula that is used to calculate the probability in response to an input of the random variable's value.

Discrete Distribution Example

- There is a chance that a bit transmitted through a digital transmission channel is received in error.
- Let X equal the number of bits received in error of the next 4 transmitted.
- The associated probability distribution of X is shown as a graph and as a table.



$P(X=0) =$	0.6561
$P(X=1) =$	0.2916
$P(X=2) =$	0.0486
$P(X=3) =$	0.0036
$P(X=4) =$	0.0001
	1.0000

Probability Mass Function (PMF)

For a discrete random variable X

with possible values $x_1, x_2, \ldots x_n$,

a probability mass function is a function such that:

$$(1) \quad f(x_i) \geq 0$$

$$(2) \quad \sum_{i=1}^{n} f(x_i) = 1$$

$$(3) \quad f(x_i) = P(X = x_i)$$

Discrete Distribution Example

- In a semiconductor manufacturing process, 2 wafers from a lot are sampled. Each wafer is classified as *pass* or *fail*. Assume that the probability that a wafer passes is 0.8, and that wafers are independent.

- The random variable X is defined as the number of wafers that pass.

Table 3-1 Wafer Tests

Outcome			
Wafer #			
1	2	x	Probability
Fail	Fail	0	0.04
Fail	Pass	1	0.16
Pass	Fail	1	0.16
Pass	Pass	2	0.64
			1.00
Probability Mass Function		0	0.04
		1	0.32
		2	0.64
			1.00

PMF Example

- Let the random variable X denote the number of wafers that need to be analyzed to detect a large particle.
- Assume that the probability that a wafer contains a large particle is 0.1, and that the wafers are independent.
- Determine the probability distribution of X.
  - Let p denote a wafer for which a large particle is present & let a denote a wafer in which it is absent. P(p) = 0.1, P(a)=0.9
  - The sample space is: $S = \{p, ap, aap, aaap, \ldots\}$
  - The range of the values of X is: x = 1, 2, 3, 4, …

Probability Distribution		
P(X=1)	0.1	0.1
P(X=2)	0.9*0.1	0.09
P(X=3)	0.9^2*0.1	0.081
P(X=4)	0.9^3*0.1	0.0729
…	…	…

- A formula (PMF)

$P(X=x) = 0.9^{x-1}(0.1)$

Cumulative Distribution Function (CDF)

- The cumulative distribution function can be built from the probability mass function and vice versa.

The cumulative distribution function of a discrete random variable X, denoted as $F(x)$, is:

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

For a discrete random variable X, $F(x)$ satisfies the following properties:

(1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

(2) $0 \leq F(x) \leq 1$

(3) If $x \leq y$, then $F(x) \leq F(y)$

CDF Example

- Going back to example from slide 6, we can express the probability of three or fewer bits being in error, denoted as $P(X \leq 3)$.

- The event $X \leq 3$ is the union of the mutually exclusive (disjoint) events: $X = 0$, $X = 1$, $X = 2$, $X = 3$.

- From the Table:
- $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = 0.9999$
- $P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.0036$

	Mass	Cumulative
x	$P(X = x)$	$P(X \leq x)$
0	0.6561	0.6561
1	0.2916	0.9477
2	0.0486	0.9963
3	0.0036	0.9999
4	0.0001	1.0000
	1.0000	

Numbers to describe our distributions

- The mean is a measure of the center of a probability distribution.
- The variance is a measure of the dispersion or variability of a probability distribution.
- The standard deviation is another measure of the dispersion. It is the square root of the variance.

Mean

The mean or expected value of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_{x} x \cdot f(x)$$

- The mean, E(X), is the:
  1. Probability-weighted average of the possible values of *X*.
  2. "Center of Mass"
  3. Most common way to characterize the center of the distribution.

- The mean value may, or may not, be a given value of *x*.

Expected Value Calculation

- From a previous example, there is a chance that a bit transmitted through a digital transmission channel is an error.

- *X* is the number of bits received in error of the next 4 transmitted. Calculate the mean

Expected Value		
X	f(x)	x*f(x)
0	0.6561	0
1	0.2916	0.2916
2	0.0486	0.0972
3	0.0036	0.0108
4	0.001	0.004
	E[X]=	0.4036

Variance of an RV

The variance of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 \cdot f(x) = \sum_x x^2 \cdot f(x) - \mu^2$$

$\mu = \sum_x x f(x)$ and $\sum_x f(x) = 1$ are truths.

$$V(X) = \sum_x (x-\mu)^2 f(x) \text{ is the definitional formula}$$

$$= \sum_x \left(x^2 - 2\mu x + \mu^2\right) f(x)$$

$$= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x)$$

$$= \sum_x x^2 f(x) - 2\mu^2 + \mu^2$$

$$= \sum x^2 f(x) - \mu^2 \text{ is the computational formula}$$

Variance in Computations

- One property of the Expected value operator is:

$$E[h(x)] = \sum h(x) * f(x)$$

- So we can write our computational definition of Variance as:

$$\sum_x x^2 f(x) - \mu^2$$

<=>

$$V[X] = E[X^2] - E[X]^2$$

Note: $E(X^2) \neq [E(X)]^2$

Standard Deviation

- To find a Standard deviation, We just take the square root of the variance

The standard deviation of X, denoted as σ is

$$\sigma = \sqrt{V(X)}$$

Example of mean and variance

- Going back again to slide 6, there is a chance that a bit transmitted through a digital transmission channel is an error. X is the number of bits received in error of the next 4 transmitted. Use a table to calculate the mean & variance.

Expected Value			Definiational Formula			Computational Formula	
X	f(x)	x*f(x)	x−μ	(x−μ)^2	(x−μ)^2*f(x)	x^2	x^2*f(x)
0	0.6561	0	-0.4	0.16	0.104976	0	0
1	0.2916	0.2916	0.6	0.36	0.104976	1	0.2916
2	0.0486	0.0972	1.6	2.56	0.124416	4	0.1944
3	0.0036	0.0108	2.6	6.76	0.024336	9	0.0324
4	0.0001	0.0004	3.6	12.96	0.001296	16	0.0016
	E[X]=	0.4		V[X]=	0.36	E[X^2]=	0.52
						V[X]=	0.36

Using V[X] $= E[X^2] - E[X]^2$

Some Special Discrete Distributions

- Sometimes we can use specific distributions to model situations

- Some important distributions we will cover are:
  - Binomial
  - Poisson

- Some other important ones to be familiar with:
  - Geometric
  - Negative Binomial
  - Hypergeometric

Binomial Distribution

- To Model a situation with the Binomial Distribution we must have:
  - Fixed number of trials (n).
  - Each trial is deemed a success or failure. (Bernoulli trials)
  - The probability of success in each trial is constant (p).
  - The outcomes of successive trials are independent.

Binomial Distribution

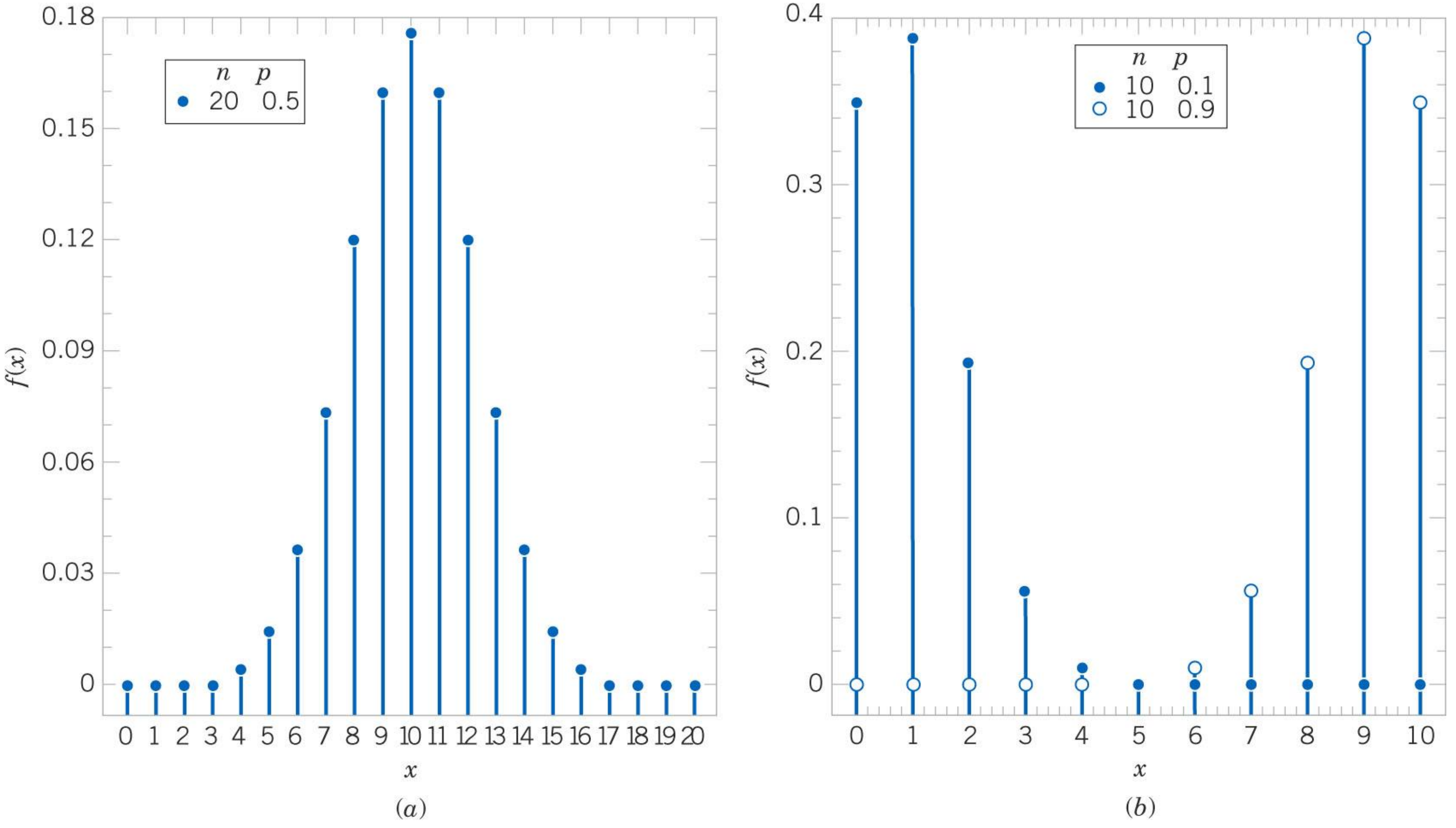
- Let X be a binomial random variable that equals the number of trials that result in a success with parameters $0 < p < 1$ and n = 0, 1, ...

- Denoted: B(n,p)

- $f(x) = C_x^n p^x (1-p)^{n-x}$ and $F(x) = \sum_{x<x_i} f(x)$

- Measures
  - μ = E[X] = n*p
  - σ^2 = V[X] = n*p(1−p)

Binomial Distribution Shapes



Ways to solve Binomial problems

- Table from your book :(
- Pencil and Paper :|
- Graphing in Minitab :)
- Excel :) :)

Binomial example

- Samples of water have a 10% chance of containing a pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

- Let X denote the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p = 0.1$ and $n = 18$.

$$P(X = 2) = C_2^{18}(0.1)^2(0.9)^{16} = 153(0.1)^2(0.9)^{16} = 0.2835$$

0.2835	=BINOMDIST(2,18,0.1,FALSE)

Binomial example cont.

- Determine the probability that at least 4 samples contain the pollutant. B(18,0.1).

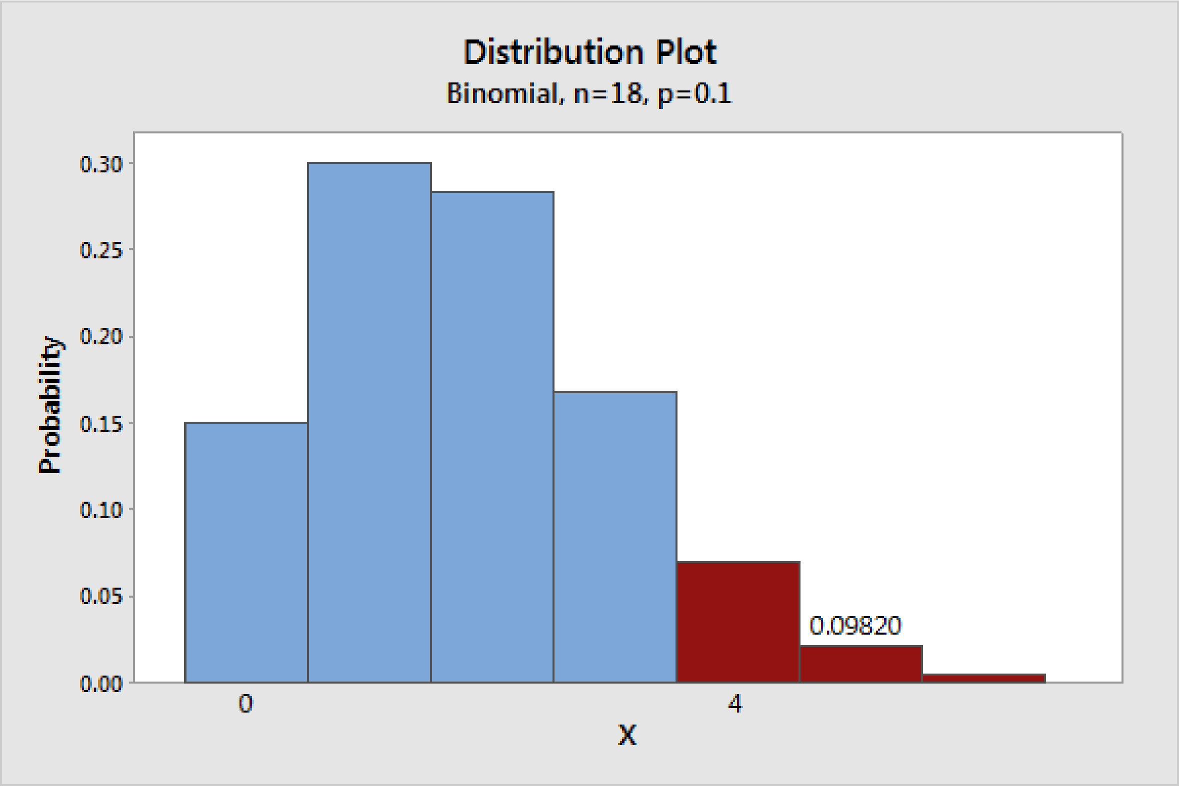
$$P(X \geq 4) = \sum_{x=4}^{18} C_x^{18} (0.1)^x (0.9)^{18-x}$$
$$= 1 - P(X < 4)$$
$$= 1 - \sum_{x=0}^{3} C_x^{18} (0.1)^x (0.9)^{18-x}$$
$$= 0.098$$

0.0982	= 1 - BINOMDIST(3,18,0.1,TRUE)

- Lets take a look at graphing this in Minitab

Binomial example cont.

- Graph of $P(X \geq 4)$ for $B(18,0.1)$
- Steps:
  - Go to graph menu → Probability Distribution Plot
  - Select Single →Choose Binomial, enter parameters
  - To select desired shaded area double click on bars of graph
  - Click on the shaded area tab and enter what you would like to shade



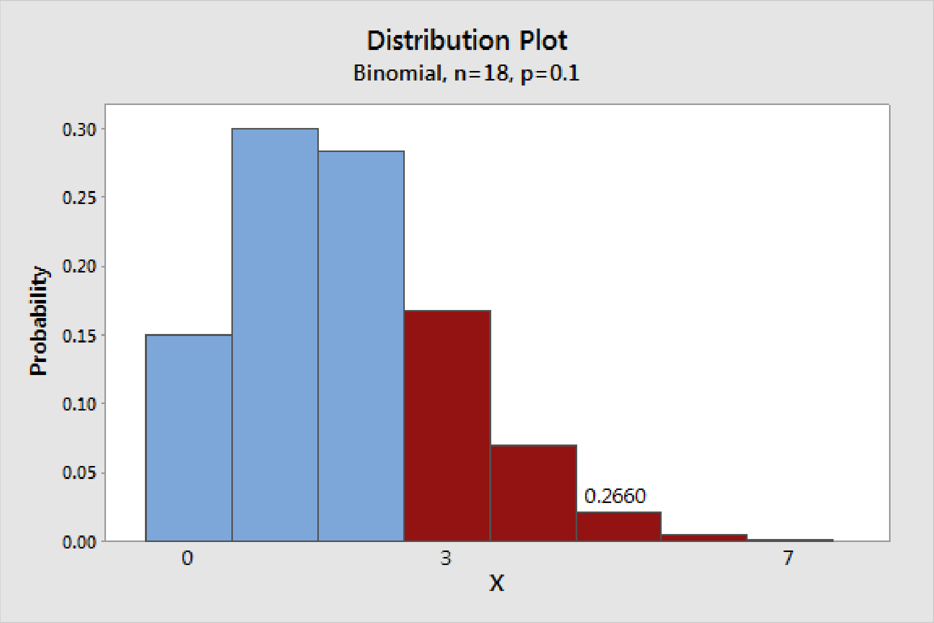
Binomial example cont.

- Now determine the probability that $3 \leq X \leq 7$. B(18,0.1)

$$P(3 \leq X \leq 7) = \sum_{x=3}^{7} C_x^{18} (0.1)^x (0.9)^{18-x} = 0.265$$

$$P(X \leq 7) - P(X \leq 2)$$

0.2660=BINOM.DIST.RANGE(18,0.1,3,7)



Transmission example Binomial

- Recall the previous example about the number of transmitted bits received in error.
- We could use: $n = 4$ and $p = 0.1$.
- Find the mean, variance & std dev of this binomial random variable

- $\mu = E(X) = np = 4*0.1 = 0.4$
- $\sigma^2 = V(X) = np(1-p) = 4*0.1*0.9 = 0.36$
- $\sigma = SD(X) = 0.60$
- Compare to earlier:

Expected Value			Definiational Formula			Computational Formula	
X	f(x)	x*f(x)	x-μ	(x-μ)^2	(x-μ)^2*f(x)	x^2	x^2*f(x)
0	0.6561	0	-0.4	0.16	0.104976	0	0
1	0.2916	0.2916	0.6	0.36	0.104976	1	0.2916
2	0.0486	0.0972	1.6	2.56	0.124416	4	0.1944
3	0.0036	0.0108	2.6	6.76	0.024336	9	0.0324
4	0.0001	0.0004	3.6	12.96	0.001296	16	0.0016
	E[X]=	0.4		V[X]=	0.36	E[X^2]=	0.52
						V[X]=	0.36

Binomial →Poisson

- As the number of trials (*n*) in a binomial experiment increases to infinity while the binomial mean (*np*) remains constant, the PMF of the binomial distribution becomes the PMF of the Poisson distribution.

$$\text{Let } \lambda = np = E(x), \text{ so } p = \lambda/n$$

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \underset{lim_{x\to\infty}}{\longrightarrow}$$

$$= \frac{e^{-\lambda}\lambda^x}{x!}$$

Poisson Distribution

- In general, the Poisson random variable X is the number of events (counts) on a fixed interval.

- Examples:
  - Particles of contamination per wafer.
  - Flaws per batch.
  - Calls at a customer service center per hour.
  - Power outages per year.

Poisson Distribution

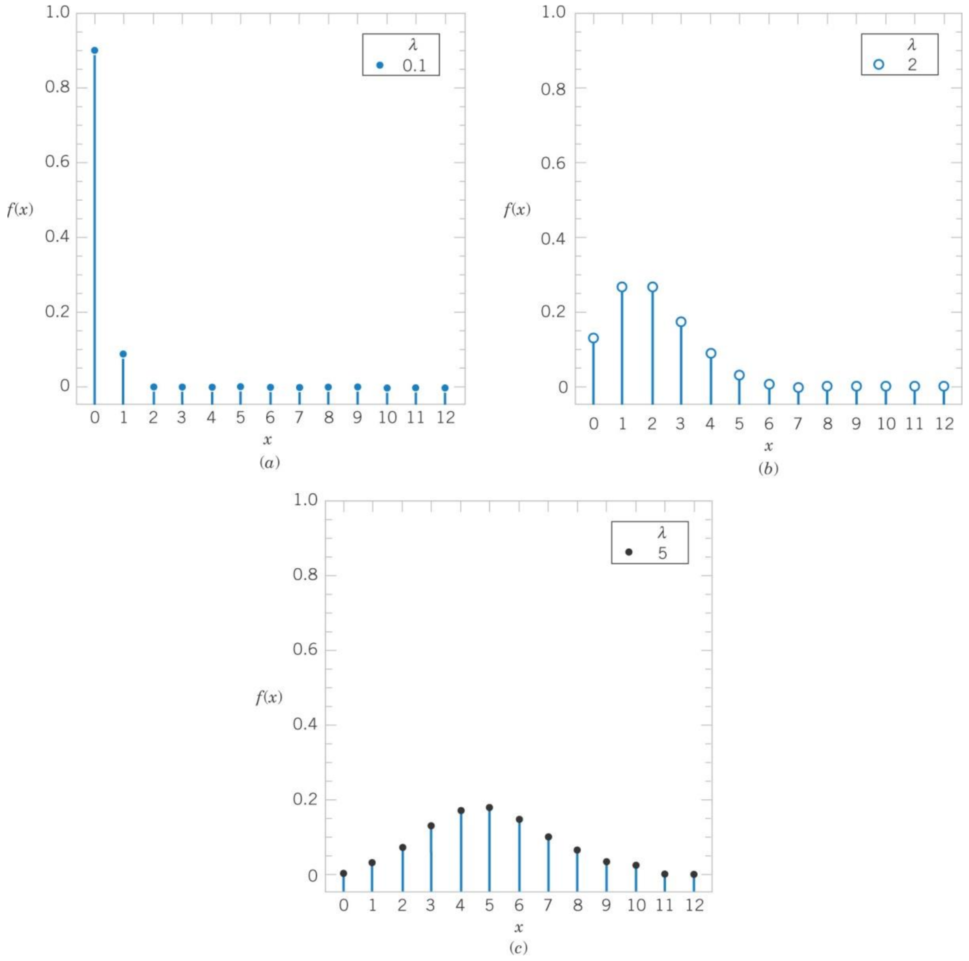
- The random variable X that equals the number of events in a Poisson process is a Poisson random variable with parameter $\lambda > 0$.

- Denoted: P(λ)

- $f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ and $F(X) = e^{-\lambda}\sum_{i=0}^{x}\frac{e^{-\lambda}}{i!}$

- Measures
  - $\mu = E[X] = \lambda = V[X] = \sigma^2$

Poisson Graphs



Poisson Example

- For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution of 2.3 flaws per mm. Let X denote the number of flaws in 1 mm of wire. Find the probability of exactly 2 flaws in 1 mm of wire.

$$P(X = 2) = \frac{e^{-2.3} 2.3^2}{2!} = 0.265$$

In Excel	
0.2652	= POISSON.DIST(2, 2.3, FALSE)

Poisson Cautions

- It is important to use consistent units in the calculation of Poisson:
  - Probabilities
  - Means
  - Variances

- Example of unit conversions:
  - Average # of flaws per mm of wire is 3.4.
  - Average # of flaws per 10 mm of wire is 34.
  - Average # of flaws per 20 mm of wire is 68.

Poisson Example cont…

- Determine the probability of 10 flaws in 5 mm of wire.

- Now, let X denote the number of flaws in 5 mm of wire.

$$E(X) = \lambda = 5 \text{ mm} \cdot 2.3 \text{ flaws/mm} = 11.5 \text{ flaws}$$

$$P(X = 10) = e^{-11.5} \frac{11.5^{10}}{10!} = 0.113$$

In Excel	
0.1129	= POISSON.DIST(10, 11.5, FALSE)

Poisson Example cont…

- Determine the probability of at least 1 flaw in 2 mm of wire.
- Now let X denote the number of flaws in 2 mm of wire.
  - Note that P(X ≥ 1) requires ∞ terms. Can't do that computationally

$$E(X) = \lambda = 2 \text{ mm} \cdot \ 2.3 \text{ flaws/mm} = 4.6 \text{ flaws}$$

$$\mathrm{P}(X \geq 1) = 1 - P(X = 0) = 1 - e^{-4.6}\frac{4.6^0}{0!} = 0.9899$$

In Excel	
0.9899	= 1 - POISSON.DIST(0, 4.6, FALSE)

Well-Known Discrete Distributions

- Poisson
  - The number of events (x) likely to happen on an interval with rate λ
- Hypergeometric
  - When drawing from a set of N items with D items of interest, what is the probability of drawing (x) items of interest in a set of n items (w/o replacement), ?
- Uniform
  - The probability of n equally likely outcomes

These distributions all deal with series of independent Bernoulli trials:

- Binomial
  - Probability of x successes in n trials
- Geometric
  - Number of trials, x, until a (1^{st}) success
- Negative Binomial
  - Numbers of trials, x, until r successes occur

Comparing Discrete Distributions

Consider a deck of card. These are the type of questions

- Uniform
  - What is the probability of drawing the Ace of Spades?
- Binomial
  - In 5 draws, with replacement, what is the probability of drawing 2 aces?
- Geometric
  - Number of draws with replacement until you get an Ace
- Negative Binomial
  - What is the probability that in your last 5 hands you have had two aces?
- Hypergeometric
  - When drawing a 5 card hand (w/o replacement), what is the probability you get a pair of Aces?
- Poisson
  - The rate of getting an ace per 100 hands is λ, what is the probability of getting 500 aces in 1000 hands?